ISRAEL JOURNAL OF MATHEMATICS, Vol. 39, No. 3, 1981

INJECTIVE AND FLAT COVERS, ENVELOPES AND RESOLVENTS

BY
EDGAR E. ENOCHS

ABSTRACT

Using the dual of a categorical definition of an injective envelope, injective covers can be defined. For a ring R, every left R-module is shown to have an injective cover if and only if R is left noetherian. Flat envelopes are defined and shown to exist for all modules over a regular local ring of dimension 2. Using injective covers, minimal injective resolvents can be defined.

1. Introduction

An injective envelope of a left R-module M can be characterized as a linear map $\phi: M \to E$ into an injective R-module E with two properties:

(a) Any diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & E \\ & \searrow & \vdots \\ & & E' \end{array}$$

where E' is an injective left R-module can be completed (or equivalently, ϕ is an injection).

(b) The diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & E \\ & {\scriptstyle \phi}\searrow & \vdots \\ & & E \end{array}$$

can be completed only by automorphisms of E (equivalently, E is an essential extension of $\phi(M)$).

Dually, an injective cover of M is a linear map $\psi: E \to M$ with E injective such that:

Received July 9, 1980

(1) Any diagram

$$\begin{array}{ccc} E' & & \\ \vdots & \searrow & \\ \downarrow & & \\ E & \xrightarrow{\psi} & M \end{array}$$

with E' injective can be completed.

(2) The diagram

$$\begin{array}{ccc} E & & \\ \vdots & \searrow^{\psi} & \\ \downarrow & & \\ E & \xrightarrow{\psi} & M \end{array}$$

can be completed only by automorphisms of E.

Note that an injective cover of M (if it exists) is unique up to isomorphism. Also note that the kernel of an injective cover $E \to M$ contains no non-zero injective modules. For any module M over a Dedekind domain, the natural embedding $E \to M$ of the largest divisible submodule E of M is the injective cover of M. If $\psi : E \to M$ satisfies (1) and perhaps not (2), it is called an injective precover. An injective preenvelope is defined similarly. By analogy, flat envelopes and covers, projective envelopes and covers, and other types of covers and envelopes can be defined. Note that although this definition of a projective cover is not the usual one [1], it is in agreement with it. One object of this paper is to show that left noetherian rings are precisely those rings for which every left module has an injective cover. The bulk of the proof consists in deducing the existence of an injective cover from the existence of a precover. Some of the proofs will apply to other types of covers and envelopes (e.g., flat covers).

2. Injective covers

We first show:

PROPOSITION 2.1. *If R is a ring such that every left R-module has an injective precover, then R is left noetherian.*

PROOF. It suffices to show that for any family (E_i), $i \in I$ of injective left R-modules, $\bigoplus E_i$ is injective. If $E \to \bigoplus E_i$ is a precover and $j \in I$ then completing

$$\begin{array}{ccc} & & E_j \\ & \swarrow^{\iota} & \downarrow \\ E & \longrightarrow & \bigoplus E_i \end{array}$$

with $E_j \to \bigoplus E_i$ the canonical injective gives rise to a map $\bigoplus E_i \to E$ with the composition $\bigoplus E_i \to E \to \bigoplus E_i$ the identity. Hence $\bigoplus E_i$ is isomorphic to a direct summand of E and so is injective.

The next result in different terms appears in Teply [8].

PROPOSITION 2.2. (Teply). *If R is left noetherian, every left R-module M has an injective precover.*

PROOF. Since R is left noetherian, there is a set X of injective left R-modules such that any injective left R-module is the direct sum of modules each isomorphic to an element of X [4]. Hence $F \to M$ with F injective will be a precover only if

$$\begin{array}{ccc} & \swarrow\!\!\cdot\cdot & E \\ & & \downarrow \\ F & \longrightarrow & M \end{array}$$

can be completed for each $E \in X$. For $E \in X$ let E^* be the sum of $\mathrm{Hom}(E, M)$ copies of E and let $E^* \to M$ map (x_ϕ), $\phi \in \mathrm{Hom}(E, M)$ onto $\Sigma\, \phi(x_\phi)$. For any map $\phi' : E \to M$,

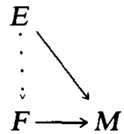
$$\begin{array}{ccc} & \swarrow\!\!\cdot\cdot & E \\ & & \downarrow \phi \\ E^* & \longrightarrow & M \end{array}$$

can be completed by mapping E onto the ϕ' component of E^*. But then $\bigoplus E^*$ (with the sum taken over $E \in X$) and the map $\bigoplus E^* \to M$ which comes from the maps above give an injective precover of M.

THEOREM 2.1. *A ring R is left noetherian if and only if every left R-module has an injective cover.*

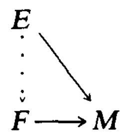
The "only if" part is Proposition 2.1, so assuming R is left noetherian, the rest of the proof will consist in using the injective precovers guaranteed by Proposition 2.2 to find injective covers. R being left noetherian is used in noting that it implies that inductive limits of injective left R-modules are injective.

We break the proof into three lemmas. The motivation for the first two lemmas is the observation that if $E \to M$ is an injective cover and $F \to M$ is a precover, then any way we complete

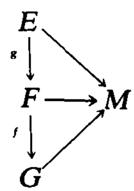


the map $E \to F$ must be an injection (compositing with the appropriate map $F \to E$ is an automorphism of E). And so, Lemmas 2.1 and 2.2 guarantee an injective precover $E \to M$ with this additional property.

LEMMA 2.1. *If $E \to M$ is a linear map where E is injective, then there exists an injective precover $F \to M$ and a map $f : E \to F$ completing*

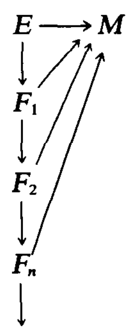


such that for any commutative diagram



where $G \to M$ is also a precover, $\ker(g \circ f) = \ker(f)$ (*i.e. the kernel of f is in some sense maximal*).

PROOF. Suppose the conclusion is not true. Then any such f doesn't have the desired property. Hence we can construct a diagram



with $\ker(E \to F_n) \subsetneqq \ker(E \to F_{n+1})$ for each $n \geqq 1$ and with each $F_n \to M$ a precover. Letting $F_\omega = \varinjlim F_n$ (ω is the first infinite ordinal) we see that F_ω is injective and in fact $F_\omega \to M$ is an injective precover. Also $\ker(E \to F_n) \subsetneqq \ker(E \to F_\omega)$ for each n. But the map $E \to F_\omega$ doesn't have the desired property so there is a precover $F_{\omega+1} \to M$ and a map $F_\omega \to F_{\omega+1}$ completing

$$\begin{array}{ccc} F_\omega & & \\ \vdots & \searrow & \\ \vdots & & M \\ \downarrow & \nearrow & \\ F_{\omega+1} & & \end{array}$$

such that $\ker(E \to F_\omega) \subsetneqq \ker(E \to F_{\omega+1})$. Continuing in this fashion we see that for any ordinal α we can construct injective precovers $F_\beta \to M$ for all $\beta < \alpha$ with maps $E \to E_\beta$ such that for $\beta < \nu < \alpha$, $\ker(E \to F_\beta) \subsetneqq \ker(E \to F_\nu)$. If for each β with $\beta + 1 < \alpha$ we choose x_β with $x_\beta \notin \ker(E \to F_\beta)$, $x_\beta \in \ker(E \to F_{\beta+1})$ then $x_\beta \neq x_{\beta'}$ for distinct β, β'. This implies $\mathrm{Card}(E) \geqq \mathrm{Card}(\alpha)$ whenever α is infinite. This is clearly impossible.

Lemma 2.2. *There exists an injective precover $E \to M$ with the property that for any commutative diagram*

$$\begin{array}{ccc} E & \longrightarrow & M \\ \downarrow & \nearrow & \\ F & & \end{array}$$

with $F \to M$ a precover, $E \to F$ is an injection.

Proof. Using Lemma 2.1 we construct a diagram

$$\begin{array}{ccc} E_1 & \longrightarrow & M \\ \downarrow & \nearrow & \uparrow \\ E_2 & & \\ \downarrow & & \\ \vdots & & \\ \downarrow & \nearrow & \\ E_n & & \\ \downarrow & & \\ \vdots & & \end{array}$$

where for each $n \geqq 1$, $E_n \to M$ is an injective precover and where

$$\begin{array}{ccc} E_n & \longrightarrow & M \\ \downarrow & \nearrow & \\ E_{n+1} & & \end{array}$$

has property guaranteed by Lemma 2.1, i.e. if

$$\begin{array}{ccc} E_n & \longrightarrow & M \\ \downarrow & \nearrow & \uparrow \\ E_{n+1} & & \\ \downarrow & \nearrow & \\ F & & \end{array}$$

is commutative for some injective precover $F \to M$ then $\ker(E_n \to E_{n+1}) = \ker(E_n \to F)$. Now let $E = \varinjlim E_n$ and let $E \to M$ come from the maps $E_n \to M$. $E \to M$ is the desired precover F, for suppose

$$\begin{array}{ccc} E & \longrightarrow & M \\ \downarrow & \nearrow & \\ F & & \end{array}$$

is commutative with $F \to M$ a precover. If $z \in E$ is in the kernel of $E \to F$, let $x \in E_m$ map onto z in $\varinjlim E_n = E$. Our assumption on $E_m \to E_{m+1}$ applied to

$$\begin{array}{ccc} E_m & \longrightarrow & M \\ \downarrow & \nearrow & \uparrow \\ E_{m+1} & & \\ \downarrow & \nearrow & \\ F & & \end{array}$$

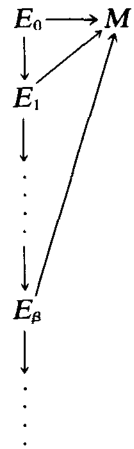
shows that $E_m \to E_{m+1}$ maps x onto 0. Consequently $z = 0$ in $\varinjlim E_n = E$.

LEMMA 2.3. *If $\psi : E \to M$ is an injective precover having the property of Lemma 2.2, then $\Psi : E \to M$ is an injective cover.*

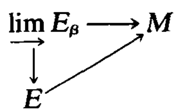
PROOF. If every map $E \to E$ completing

$$\begin{array}{ccc} E & \longrightarrow & M \\ \vdots & \nearrow & \\ E & & \end{array}$$

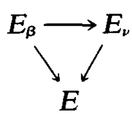
is an isomorphism, then we are through, so suppose $E \to E$ is such a map, but not an isomorphism. Then it is easy to see that for α any ordinal number we can construct a commutative diagram



where each $E_\beta = E$ and where if $\beta + 1 < \alpha$ then $E_\beta \rightarrow E_{\beta+1}$ is not an isomorphism (but is an injection). Now complete



By construction, if $\beta < \nu < \alpha + 1 < \alpha$ then



consists of injections none of which are surjections. This implies $\mathrm{Card}(E) \geqq \mathrm{Card}(\alpha)$ for all infinite α, so gives a contradiction.

Examples. Tom Cheatham and myself have argued that if R is local with residue field k, the injective cover of $E(k)/k$ has the form $E(k)^n$ for a finite $n \geqq 1$. $n > 1$ can occur but if $R = k[[x, y]]$, $n = 1$. This uses Northcott's description of $I(k)$ as the inverse polynomial ring $k[x^{-1}, y^{-1}]$ in [5].

3. Flat covers

The three lemmas of Section 2 allow one to conclude the existence of an injective cover from that of a precover if R is left noetherian. If we replace injective by flat in the three lemmas, then since the inductive limit of flat left R-modules is flat for any ring R, we get:

THEOREM 3.1. *For any ring R and any left R-module M, if M has a flat precover then it has flat cover.*

It is not known whether flat precovers always exist. If R is a domain then every module has a torsion free cover [3], hence if R is furthermore Prüfer (so flat = torsion free), flat covers exist. It seems reasonable to conjecture that they exist for any ring.

By Lazard's thesis every flat module is the inductive limit of projective modules over some directed set I. If, for a given ring R there is a "universal" I such that every flat module over R is the inductive limit of projective modules over I, then it can be shown that all left R-modules have flat precovers, so they have covers. For example, if $R = Z$, then $I = N$ (with the usual order) works. If R is a Dedekind domain, there is such an I, but its structure may be more complicated.

Note that for any left R-module M, $P \to M$ for a projective module P is a projective precover of M if and only if $P \to M$ is surjective. If R is such that flat left R-modules are projective (i.e. R is left perfect) we get as a consequence of Theorem 3.1

COROLLARY (Bass). *If all flat left R-modules are projective, then every left R-module has a projective cover.*

4. Sums of covers and envelopes

A direct sum of covers may fail to be a precover, or it may be a precover and still not be a cover. Namely, if for each $i \in I$, $\psi_i : E_i \to M_i$ is a cover, it may be possible to complete

$$\begin{array}{ccc} \bigoplus E_i & & \\ \vdots & \searrow & \\ \downarrow & & \bigoplus M_i \\ & \nearrow & \\ \bigoplus E_i & & \end{array}$$

by a map which is not an isomorphism. Our first two propositions show when this property of covers and envelopes is preserved by countable sums. The necessary condition is a sort of T-nilpotency, which when applied to projective covers gives the usual T-nilpotency of the radical of a left perfect ring and which when applied to injective envelopes in the case of a commutative noetherian ring gives another familiar result.

In the situation above, there is no loss in generality in assuming each M_i is a quotient of E_i, say E_i/S_i, and that $\psi_i : E_i \to E_i/S_i$ is the canonical surjection.

PROPOSITION 4.1. *If for each* $i = 1, 2, 3, \cdots$, $S_i \subset E_i$ *is a submodule such that*

$$\begin{array}{ccc} E_i & \searrow & \\ \vdots & & E_i/S_i \\ \downarrow & \nearrow & \\ E_i & & \end{array}$$

can be completed only by automorphisms of E_i, *then the same is true of*

$$(*) \qquad \begin{array}{ccc} \bigoplus E_i & \searrow & \\ \vdots & & \bigoplus E_i / \bigoplus S_i \\ \downarrow & \nearrow & \\ \bigoplus E_i & & \end{array}$$

if and only if for each sequence $1 \leqq k_1 < k_2 < \cdots$ *of positive integers and maps* $f_n : E_{k_n} \to E_{k_{n+1}}$ *with* $\operatorname{im}(f_n) \subset S_{k_{n+1}}$ *and for each* $x \in E_{k_1}$ *there is an* $m \geqq 1$ *such that* $f_m \circ f_{m-1} \circ \cdots \circ f_1(x) = 0$.

PROOF. We argue the necessity and let the k_n's and f_n's be as stated. Define $\phi : \bigoplus E_i \to \bigoplus E_i$ so that if $i \neq k_n$ for all n then $\phi \mid E_i$ is the identity map and so that $\phi \mid E_{k_n}$ agrees with the map $E_{k_n} \to E_{k_n} \oplus E_{k_{n+1}}$ which takes y to $(y, -f_n(y))$. Then ϕ completes the diagram (*). Furthermore one checks that if $x \in E_{k_1}$ and if x is in the image of ϕ, say $\phi((x_i)) = x$, then $x_i = 0$ for $i \neq k_n$ for all n. Also x_{k_1} must be x and by induction we see that $x_{k_n} = f_{n-1} \circ \cdots \circ f_1(x)$ for $n > 1$. But $(x_i) \in \bigoplus E_i$ implies $x_{k_n} = 0$ for n sufficiently large.

For the converse, suppose ϕ completes (*). Use the matrix notation $\phi = (\phi_{ij})$ with $\phi_{ij} : E_j \to E_i$. Note that for each i, ϕ_{ii} completes

$$\begin{array}{ccc} E_i & \searrow & \\ \vdots & & E_i/S_i \\ \downarrow & \nearrow & \\ E_i & & \end{array}$$

and so is an isomorphism, and that for $i \neq j$, ϕ_{ij} has its image in S_j. Also (ϕ_{ij}) is locally column finite in the sense that for any j and any $x \in E_j$, $\phi_{ij}(x) = 0$ except for a finite number of i. Furthermore any collection of ϕ_{ij}'s satisfying these conditions gives a ϕ completing (*). To argue that ϕ is an isomorphism we only need find a ψ which is an isomorphism completing (*) and such that $\psi \circ \phi$ or $\phi \circ \psi$ is an isomorphism. The argument proceeds by showing that ϕ has a triangular decomposition, i.e. it is the product of an upper and lower triangular matrix (corresponding to an automorphism of $\bigoplus E_i$). If ϕ is upper triangular, then since its diagonal elements are automorphisms of the E_i, it's a standard argument that it is invertible, and clearly its inverse satisfies the conditions above, so guaranteeing that it corresponds to an automorphism of $\bigoplus E_i$ of the desired type (i.e. making (*) commutative).

So we construct an upper triangular matrix ψ of the desired form so that $\phi \circ \psi$ is lower triangular. ψ will be defined as an infinite product $\psi_1 \circ \psi_2 \circ \psi_3 \circ \cdots$. Let

$$\psi_1 = \begin{bmatrix} -\phi_{11}^{-1} & -\phi_{11}^{-1}\phi_{12} & -\phi_{11}^{-1}\phi_{13} & \cdots \\ 0 & \mathrm{id} & 0 & \cdots \\ 0 & 0 & \mathrm{id} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

Then $\phi \circ \psi_1$ has the form

$$\begin{bmatrix} \mathrm{id} & 0 & 0 & \cdots \\ \phi'_{21} & \phi'_{22} & \phi'_{23} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

hence define ψ_2 as we defined ψ_1 but using the second row of $\phi \circ \psi_1$ and then similarly defining $\psi_3, \cdots$ it is easy to see that the ij entry of $\psi_1 \circ \cdots \circ \psi_n$ is constant for n sufficiently large and so the infinite product converges; and if ψ is this product it gives the desired automorphism of $\bigoplus E_i$ so that $\phi \circ \psi$ is lower triangular.

Now assume that ϕ is lower triangular and that it has the identities id_{E_i} on the diagonal. So we have $\phi = \mathrm{id}_{\oplus E_i} - K$ where K is strictly lower triangular and $-K$ has ϕ_{ij} for its ij entry when $i > j$. Since K is strictly lower triangular the sum

$$\phi' = \mathrm{id} + K + K^2 + \cdots + K^n + \cdots$$

is well defined. As a matrix ϕ' is ϕ^{-1}. However we need to argue that it is locally column finite. To argue this, given i, let $x \in E_i$. The ji entry of $\phi'(x)$ for $j > i$ is

$$\sum \phi_{jk_s} \circ \phi_{k_s k_{s-1}} \circ \cdots \circ \phi_{k_2 k_1} \circ \phi_{k_1 i}(x)$$

with the summation taken over all possible finite sequences $j > k_s > \cdots > k_1 > i$. If for an infinite number of j with $j > i$ the sum is non-zero, an easy application of the König graph theorem allows us to choose $k_1 < k_2 < k_3 < \cdots$ with $\phi_{k_m k_{m-1}} \circ \cdots \circ \phi_{k_1 i}(x) \neq 0$ for all m. Letting $f_n = \phi_{k_n k_{n-1}}$ for $n \geqq 2$ and $f_1 = \phi_{k_1 i}$ we contradict our hypothesis.

Using a similar argument we get

Proposition 4.2. *If for $i = 1, 2, 3, \cdots$, $S_i \subset E_i$ is a submodule such that*

$$\begin{array}{ccc} & \nearrow & E_i \\ S_i & & \vdots \\ & \searrow & E_i \end{array}$$

can only be completed by isomorphisms, then the same is true of

$$\begin{array}{ccc} & \nearrow & \bigoplus E_i \\ \bigoplus S_i & & \vdots \\ & \searrow & \bigoplus E_i \end{array}$$

if and only if for any sequence $1 \leqq k_1 < k_2 < \cdots$ *and maps* $f_n : E_{k_n} \to E_{k_{n+1}}$ *such that* $f_n(S_{k_n}) = 0$ *and for any* $x \in E_{k_1}$, *there is an* $m \geqq 1$ *such that*

$$f_m \circ f_{m-1} \circ \cdots \circ f_1(x) = 0.$$

REMARK. We note that the finite counterpart of Proposition 4.1 (respectively 4.2) holds with the only hypothesis being that

$$\begin{array}{ccc} E_i & \searrow & \\ \vdots & & E_i/S_i \\ E_i & \nearrow & \end{array} \quad \left[\text{respectively} \quad \begin{array}{ccc} & \nearrow & E_i \\ S_i & & \vdots \\ & \searrow & E_i \end{array} \right]$$

can be completed only by automorphisms for $i = 1, 2, \cdots, n$ since we can let $E_k = 0$ for $k > n$.

COROLLARY 1. *If for any set, I,* $S_i \subset E_i$ *is a submodule such that*

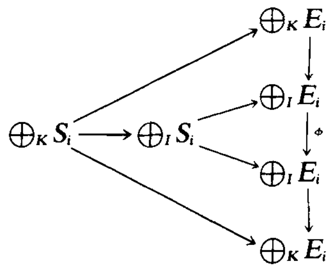
$$\begin{array}{ccc} & \nearrow & E_i \\ S_i & & \vdots \\ & \searrow & E_i \end{array}$$

can be completed only by automorphisms, then

$$\begin{array}{ccc} & \nearrow & \bigoplus E_i \\ \bigoplus S_i & & \vdots \\ & \searrow & \bigoplus E_i \end{array}$$

can be completed only by injection.

PROOF. If ϕ completes the above, then for any finite subset $K \subset I$ consider



By the remark, the vertical composition is an isomorphism. But since this is true for any finite $K \subset I$, ϕ must be an injection.

COROLLARY 2. *If for each $i \in I$, $M_i \to F_i$ is a flat envelope and if $\bigoplus M_i$ has a flat envelope, then $\bigoplus M_i \to \bigoplus F_i$ is a flat envelope.*

PROOF. Assume $\bigoplus M_i \to F$ is a flat envelope and choose the obvious diagrams.

COROLLARY 3. *For a left perfect ring R, if $P_i \to M_i$ for each $i \in I$ are projective covers of left R-modules, then $\bigoplus P_i \to \bigoplus M_i$ is a projective cover.*

PROOF. Similar to that for Corollary 2.

REMARK 1. If we apply this to a countable sum of copies of the projective cover $R \to R/J$ where J is the Jacobson radical of R we get that $\bigoplus R \to \bigoplus R/J$ (countable sum) is a projective cover. If $r_1, r_2, \cdots$ is a sequence of elements of J, then use Proposition 4.1 letting $k_n = n$ for all n and letting $f_n : R \to R$ be multiplication by r_n. Then the condition $f_m \circ \cdots \circ f_1(x) = 0$ for $x = 1 \in R$ gives $r_m \cdots r_1 = 0$. Thus J is right T-nilpotent.

REMARK 2. If $M \subset E$ is an injective envelope of M over the commutative noetherian ring R and $rM = 0$ for some $r \in R$ then using a similar argument but letting $f_n : E \to E$ be multiplication by r for each n we get that for each $x \in E$, $r^n x = 0$ for some $n \geqq 1$. If $I \subset R$ is an ideal and $IM = 0$, it is easy to deduce that $I^n x = 0$ for some $n \geqq 1$.

5. Flat preenvelopes

Flat envelopes don't always exist, nor do flat preenvelopes. To find an example of a module not having a flat envelope we use

LEMMA 5.1. *If $M \to F$ is a flat envelope and M is finitely presented, then F is finitely generated and projective.*

PROOF. Since F is flat and M finitely presented, the map $M \to F$ can be factored through a finitely generated projective P. If $M \to P \to F$ is such a factorization then complete

$$\begin{array}{ccc} M & \longrightarrow & F \\ & \searrow & \vdots \\ & & P \end{array}$$

The composition $F \to P \to F$ must be an isomorphism, so the result follows.

COROLLARY. *If R is a domain, $M \xrightarrow{\phi} F$ is as above, and if the sum of countably many copies of M has a flat envelope, then the rank of M equals the rank of F.*

PROOF. If $\operatorname{rank} M < \operatorname{rank} F$, then $f/\phi(M)$ has a rank 1 torsion free quotient say $F/\phi(M)/F'/\phi(M) \cong F/F'$. If $x \in F$, $x \neq 0$, there is an injection $F/F' \to Rx$. If $x \in F$, $x \notin F'$, let $f: F \to F$ be the composition $F \to F/F' \to Rx \to F$, so that $f(x) = rx$ with $r \neq 0$ and $f(\phi(M)) = 0$. By Proposition 4.2 with each $f_m = f$ we should get $f \circ \cdots \circ f(x) = 0$ where f is repeated some m times. This means $r^m x = 0$ which is impossible.

If R is a local domain and $I \subset R$ a finitely presented ideal, then it is an easy argument to show that I has a flat envelope of the same rank if and only if there is a smallest principal ideal (r) containing I with the property that $I^{-1}r \subset R$. In this case $I \to (r)$ is the flat envelope. An example is $I = (x, y) \subset k[[x, y]]$ with k a field.

PROPOSITION 5.1. *For a ring R, every left R-module has a flat preenvelope if and only if R is coherent.*

PROOF. For the "only if" let (F_i), $i \in I$ be a family of flat left R-modules. If ΠF_i has a flat enevelope, then an argument dual to that for Proposition 2.1 shows that ΠF_i is flat.

Conversely, if R is any ring and $\aleph_\alpha$ is an infinite cardinal, there is an infinite cardinal $\aleph_\beta$ such that if S is a submodule of a flat module F with $\operatorname{Card}(S) \leqq \aleph_\alpha$, there is a pure, hence flat, submodule G of F with $S \subset G$ and $\operatorname{Card}(G) \leqq \aleph_\beta$. This observation means that if M is any left R-module with $\operatorname{Card}(M) \leqq \aleph_\alpha$, any homomorphism $M \to F$ with F flat can be "cut down" to a homomorphism $M \to G$, $G \subset F$, $\operatorname{Card}(G) \leqq \aleph_\beta$, G flat, which agrees with the original. Setting two such homomorphisms $M \to G$, $M \to G'$ equivalent if

$$\begin{array}{ccc} & & G \\ & \nearrow & \vdots \\ M & & \vdots \\ & \searrow & \downarrow \\ & & G' \end{array}$$

can be completed by an isomorphism, and letting X be a set of representatives of such $M \to G$, $M \to \Pi G$ will be a flat preenvelope if R is right coherent, since then ΠG is flat.

6. Flat envelopes

THEOREM 6.1. (i) *For a domain R the following are equivalent*:

(a) *every R-module has a flat envelope,*

(b) *the projective limit of any projective system of flat modules is flat.*

(ii) *Conditions* (a), (b) *above imply*

(c) *the weak global dimension of R is less than or equal to* 2.

(iii) *If R is noetherian and local then* (a), (b) *are equivalent to* (c).

REMARK. (a), (b) and (c) fail to be equivalent for an arbitrary ring, with $Z/(4)$ an easy counterexample (injective modules are flat so (a) holds but (c) fails to hold). (b) and (c) are shown to be equivalent in the context of functors in Oberst and Röhrl [6].

PROOF. (a) $\Rightarrow$ (b). Using Proposition 5.1 we get that R is coherent, so the product of flat modules is flat. To get (b) it suffices to show that the intersection of a collection of flat submodules of a flat module is flat. But note that if we have an inductive limit $\varinjlim M_i$ of finitely presented modules and if $M_i \to F_i$ is a flat envelope for each i, then by the Corollary to Lemma 5.1, the rank of M_i is the rank of F_i. Hence for any flat (hence torsion free) module F

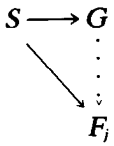
$$\begin{array}{ccc} M_i & \longrightarrow & F_i \\ & \searrow & \vdots \\ & & F \end{array}$$

can be completed uniquely. This implies that we can form $\varinjlim F_i$ and in fact that $\varinjlim M_i \to \varinjlim F_i$ is a flat envelope with the additional property that

$$\begin{array}{ccc} \varinjlim M_i & \longrightarrow & \varinjlim F_i \\ & \searrow & \vdots \\ & & F \end{array}$$

can be completed uniquely for any flat module F. For since if K is the field of fractions of R, $K \otimes \varinjlim M_i \simeq \varinjlim (K \otimes M_i) \simeq \varinjlim (K \otimes F_i) \simeq K \otimes \varinjlim F_i$.

Since R is a coherent domain, every submodule S of a flat module F is the directed union of finitely generated, hence finitely presented, submodules. This means that the flat envelope of S, say $S \to G$ will have the unique mapping property above. Now suppose $S = \bigcap F_i$ where (F_i) is some collection of flat submodules of F. Completing

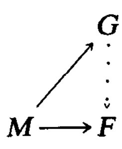


by a unique injection for each j, we see that the image of G in any F_j is in fact in $\bigcap F_i$ and so S is the image of G and so is flat.

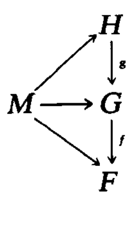
(b) $\Rightarrow$ (a). The argument is dual to the proof of the "if" part of Theorem 2.1 with the exception of Lemma 2.2. In this lemma the fact that if a map $\varinjlim E_n \to M$ is such that $E_n \to M$ and $E_n \to E_{n+1}$ have the same kernels then $\varinjlim E_n \to M$ is an injection. For modules the dual fails. We can have a non-surjective map $M \to \varprojlim E_n$ with $M \to E_n$ a surjection for each n. Hence our argument must be a little more subtle. The argument and result may have independent interest and applications to other situations.

PROPOSITION 6.1. *If R is right coherent and any projective limit of flat left R-modules is flat then every module has a flat envelope.*

PROOF. By Proposition 5.1 we know every left R-module has a flat preenvelope. By the argument dual to that of Lemma 2.1, we know that if $M \to F$ is a flat preenvelope, that there exists a diagram



with $M \to G$ a flat preenvelope such that if $M \to H$ is any other flat preenvelope and



is commutative, then $\mathrm{im}(f \circ g) = \mathrm{im}(f)$ (so in this case the image of f is minimal). We now try to find a flat preenvelope $M \to F$ such that if $M \to K$ is any other preenvelope and we complete

$$\begin{array}{ccc} & & K \\ & \nearrow & \vdots \\ M & \longrightarrow & F \end{array}$$

we have $K \to F$ is a surjection. If this is not the case, letting α be an infinite ordinal we construct a projective system

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ & & F_\beta \\ & \nearrow & \downarrow \\ & & F_1 \\ & \nearrow & \downarrow \\ M & \longrightarrow & F_0 \end{array}$$

for $\beta < \alpha$ where for each $\beta + 1 < \alpha$ we suppose that the image of $F_{\beta+1} \to F_\beta$ is minimal in the sense above. Then for each $\beta + 1 < \alpha$ let $U_\beta \subset F_\beta$ be this image. Let $M \to F$ be any flat preenvelope of M. Then complete

$$\begin{array}{ccc} & & F \\ & \swarrow & \vdots \\ M & \longrightarrow & \varprojlim F_\beta \end{array}$$

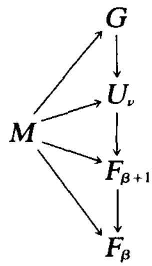
The map $F \to F_\beta$ can be factored $F \to U_\beta \to F_\beta$ when $\beta + 1 < \alpha$. By our hypothesis, $F \to U_\beta$ is a surjection. Now if $\beta < \nu < \nu + 1 < \alpha$ consider

$$\begin{array}{ccc} & F & \\ \swarrow & & \searrow \\ U_\nu & \longrightarrow & U_\beta \end{array}$$

Then $U_\nu \to U_\beta$ is a surjection. If it is never an isomorphism, then $\ker(F \to U_\nu) \supsetneqq \ker(F \to U_\beta)$. This implies $\mathrm{Card}(F) \geqq \mathrm{Card}(\alpha)$ for any such α which is impossible. Hence $U_\alpha \to U_\beta$ is an isomorphism for some such $\beta < \nu < \nu + 1 \leqq \alpha$. This map is a composition $U_\nu \to F_{\beta+1} \to U_\beta$ so that $U_\nu \simeq U_\beta$ is a retract of $F_{\beta+1}$ and so is flat. $M \to U_\nu$ is then clearly a flat preenvelope. If

$$\begin{array}{ccc} & & G \\ & \swarrow & \downarrow \\ M & \longrightarrow & U_\nu \end{array}$$

is commutative with $M \to G$ a flat preenvelope then consider



By our hypothesis on $F_{\beta+1} \to F_\beta$, the image of $G \to F$ is U_β. Since $U_\nu \to U_\beta$ is an isomorphism this means $G \to U_\nu$ is a surjection and we have the desired flat preenvelope.

Now apply the argument dual to that for Lemma 2.3 and the result follows.

Suppose now that (a) and (b) hold. Let $S \subset F$ be a submodule of a flat module and let

$$0 \to T \to G \to S \to 0$$

be exact with G flat. We want to show that T is flat. Assuming $T \subset G$, we see by the proof (a) $\Rightarrow$ (b) that there is a flat submodule $H \subset G$ with $T \subset H$ so that $T \to H$ is a flat envelope, and that $K \otimes T \to K \otimes H$ is an isomorphism. But then H/T is torsion and is isomorphic to a submodule of S which is torsion free. Hence $T = H$ and T is flat.

If R is noetherian, then the weak global dimension is the global dimension. So it remains to show that (a) holds if the global dimension is at most 2. But (a) will hold if every finitely generated module has a flat envelope of the same rank. Let S be finitely generated. We can suppose that S is torsion free and that $S \subset F$ where F is a finitely generated free module of the same rank as S. We first show that there is a projective submodule P of F containing S and contained in any other such submodule. Note that if $S \subset P_1, P_2$ with $P_1, P_2 \subset F$ projective submodules then $S \subset P_1 \cap P_2 \subset F$ and $P_1 \cap P_2$ is projective since the global dimension of R is at most 2. We argue then that a decreasing sequence $P_1 \supset P_2 \supset \cdots$ of projective submodules of F containing S terminates. R is local so each P_n is free.

Given $P \supset P'$ with P, P' finitely generated free modules, choosing bases, let A be the matrix of the inclusion $P' \subset P$. Then A is uniquely determined up to multiplication by a unit, is non-zero, and is a unit if and only if $P = P'$. If $P \subset P' \subset P''$ with P'' also finitely generated and free and if B is the matrix for $P' \subset P''$ then BA can be used for $P \subset P''$. In the situation above, let A_n be the matrix for $P_{n+1} \subset P_n$. Since S has the same rank as F, we can find $P \subset S$ free of the rank of S. If C is the matrix for $P \subset P_1$ then $(\det C) \subset (\det A_1) \cdot \circ\circ\circ \cdot (\det A_n)$ for every n. But a regular local ring is a UFD and so the only way the above is possible is for $\det A_n$ to be a unit for all $n \geqq m$ for some m. This means $P_m = P_{m+1} = \cdots$. Hence we have the required submodule P. To show $S \subset P$ is a flat envelope, suppose we have a diagram

$$\begin{array}{ccc} S & \longrightarrow & P \\ & \searrow & \vdots \\ & & F \end{array}$$

to be completed with F flat. By attaching a direct summand to F we can assume $S \to F$ is an injection and so that $S \subset F$. But $F \to K \otimes F$ is an injection and $K \otimes F$ is injective, so we can complete

$$\begin{array}{ccc} S & \longrightarrow & P \\ & \searrow & \vdots \\ & & K \otimes F \end{array}$$

The map $P \to K \otimes F$ is an injection so enlarging $F \subset K \otimes F$ to some free submodule F', $F \subset F' \subset K \otimes F$ we can assume P and F both are submodules of F'. Then $S \subset P \cap F$ and $P \cap F$ is projective by construction of P, $P \cap F = P$ or $P \subset F$. This means

$$\begin{array}{ccc} S & \longrightarrow & P \\ & \searrow & \vdots \\ & & F \end{array}$$

can be completed.

Remark. It would be interesting to have a characterization of rings all of whose left R-modules have flat envelopes. All von-Neumann regular and all quasi-Frobenius rings are such rings.

7. A categorical version

Since the proof of the "only if" part of Theorem 2.1 is categorical it seems worthwhile stating the necessary categorical assumptions. For the most generality, the proof of Lemma 2.2 should be replaced by the argument dual to that in the proof of Proposition 6.1. Thus we avoid unnecessary assumptions on inductive limits.

$\mathcal{U}$ will be a Grothendieck universe with $N \in \mathcal{U}$. Let $\mathcal{C}$ be a category which is locally small and cosmall relative to $\mathcal{U}$. Suppose very homomorphism $A \to B$ has a natural decomposition $A \to Z \to B$ with $A \to Z$ an epimorphism and $Z \to B$ a monomorphism where natural means that if $A' \to Z' \to B'$ is such a decomposition for morphism $A' \to B'$ then any commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & Z & \longrightarrow & B \\ \downarrow & & \vdots & & \downarrow \\ A' & \longrightarrow & Z' & \longrightarrow & B' \end{array}$$

can be completed to a commutative diagram. Also suppose that every bimorphism is an isomorphism. Then if $\mathcal{F}$ is any class of objects of $\mathcal{C}$ we have Theorem. If an object A of $\mathcal{C}$ has an $\mathcal{F}$-precover and if $\mathcal{F}$ is closed under well-ordered inductive limits (i.e. the limit exists in $\mathcal{C}$, and is an object in $\mathcal{F}$) then A has a cover.

We note that there is a dual theorem for envelopes. Possible applications might be in the category of topological spaces. For example, in the category of compact topological spaces the class of connected spaces is closed under projective limits. This implies that every compact space having a connected preenvelope has an envelope. But using the long lines (for arbitrary ordinals) and a cardinality argument, a two point space can be shown not to have a connected preenvelope. Then it quickly follows that only connected compact spaces have connected envelopes. This is in marked contrast to the case of compact abelian groups where every such group has a connected envelope [3].

If for compact spaces we consider those X which can be written as the projective limits of AR's, X need not be an AR but does have a sort of injective property. If such an X is a subspace of Z and $X \to Y$ is continuous where Y is an ANR, then there is a continuous extension $Z \to Y$. It would be interesting to know if envelopes exist in this case.

8. Injective resolvents

An injective resolution of a left R-module M is an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots$ with the E^n's injective. Two such resolutions give rise to homotopically isomorphic complexes. If we attempt to find an exact sequence $\cdots \to E_1 \to E_0 \to M \to 0$ with the E_n's injective we fail. In general there is no surjection $E \to M$ with E injective. However, if R is left noetherian, we can construct such a complex with the requirement that $E_0 \to M$, $E_1 \to \ker(E_0 \to M)$ and $E_{n+1} \to \ker(E_n \to E_{n-1})$ for $n \geqq 1$ are injective precovers. This means that if

E is injective and we apply $\mathrm{Hom}(E, \_\_)$ to the complex, we get an exact sequence. In this section we note some of the easy consequences of the above. We note that if we think of our category of left R-modules as a category with models, in this case the injective modules, the method of a cyclic model used in algebraic topology is applicable.

DEFINITION. The complex $\cdots \to E_1 \to E_0 \to M \to 0$ as above is called an injective resolvent of M.

Standard arguments give immediately that two injective resolvents give homotopically isomorphic complexes. If the maps $E_{n+1} \to \ker(E_n \to E_{n-1})$ are furthermore injective covers, then we call our sequence a minimal injective resolvent. The minimal injective resolvent is unique up to isomorphism, and finite sums of minimal resolvents are minimal resolvents. For $n \geqq 1$, the kernel of $E_n \to E_{n-1}$ must be an essential submodule. Otherwise a non-zero injective submodule of E_n would be mapped injectively into E_{n-1} so $E_{n-1} \to E_{n-2}$ (or $E_0 \to M$ if $n = 1$) would have a non-zero injective module in its kernel. We have

PROPOSITION 8.1. *If R is left noetherian the least n such that for the minimal resolvent of each left R-module M, $E_m = 0$ for $m > n$ is two less than the left global dimension of R, or is 0 if the global dimension is 1 or 0.*

PROOF. The argument proceeds by noting that we can use injective resolvents of M to get derived functors of $\mathrm{Hom}(N, M)$. Denote the nth derived functor as $\tilde{\mathrm{Ext}}^n(N, M)$. The n described above is the least n such that $\tilde{\mathrm{Ext}}^m(M, N) = 0$ for all N and M and all $m > n$. If $0 \to N \to G^0 \to G^1 \to \cdots$ is an injective resolution of N, then by constructing and choosing the obvious diagram, it is seen that $\tilde{\mathrm{Ext}}(N, M)$ can be computed using either the injective resolvent of M or the injective resolution of N. If the resolution of N is minimal then we see that $\tilde{\mathrm{Ext}}^n(N, G^n/\mathrm{im}(G^{n-1} \to G^n_m)) = 0$ implies $G^{n+2} = 0$. Conversely if $G^{n+2} = 0$, so that $0 \to N \to G^0 \to \cdots \to G^{n+1} \to 0$ is our resolution then since $0 \to \mathrm{Hom}(G^{n+1}, M) \to \mathrm{Hom}(G^n, M) \to \mathrm{Hom}(G^{n-1}, M)$ is exact we get $\tilde{\mathrm{Ext}}^n(N, M) = \tilde{\mathrm{Ext}}^{n+1}(N, M) = 0$. This completes the proof.

As an example of a consequence of the above, let $R = k[[x, y]]$ with K a field. If $E \to M$ is an injective cover of the R-module M with kernel S, then $\mathrm{Hom}(G, S) = 0$ for any injective module G. Otherwise the minimal injective resolvent of M would have a non-zero term of degree 1.

If R is commutative and noetherian, the E_n's in a minimal resolvent of M can be described by finding, for each prime ideal $P \subset R$, how many components of E_n are isomorphic to $E(R/P)$ (the injective hull of R/P) in a decomposition of E

into indecomposable injective modules. In Bass [2], using a minimal injective resolution instead of a resolvent, the number described is called $\mu_n(P, M)$. So for a resolvent the number will be denoted $\nu_n(P, M)$. In a subsequent publication it will be shown that, analogous to Bass' result, $\nu_n(P, M) = \dim \tilde{\text{Ext}}^n(R/P, M)_p$ where the dimension is over the field of fractions of R/P. The proof is somewhat different, largely because unlike minimal injective resolutions, minimal injective resolvents aren't preserved by localizations.

$\tilde{\text{Ext}}(N, M)$ has long exact sequences for short exact sequences $0 \to N' \to N \to N'' \to 0$ and for sequences $0 \to M' \to M \to M'' \to 0$ which may not be exact, but which become exact when we apply $\text{Hom}(E, \_\_)$ for any injective module E.

References

1. H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

2. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.

3. E. Enochs, *Torsion free covering modules*, Proc. Amer. Math. Soc. **14** (1963), 884–889.

4. P. Gabriel, *Objects injectifs dans les Catégories abéliennes*, Sém. Dubreil, 1958/59.

5. D. G. Northcott, *Injective envelopes and inverse polynomials*, J. London Math. Soc. **68** (1974), 290–296.

6. U. Oberst and H. Röhrl, *Flat and coherent functors*, J. Algebra **14** (1970), 91–105.

7. R. Switzer, *Algebraic Topology — Homotopy and Homology*, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

8. M. Teply, *Torsion-free covers II*, Israel J. Math. **23** (1976), 132–135.

Department of Mathematics
University of Kentucky
Lexington, KY 40506 USA